Constraints and symmetry in mechanics of affine motion

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Abstract

The aim of this paper is to perform a deeper geometric analysis of problems appearing in dynamics of affinely rigid bodies. First of all we present a geometric interpretation of the polar and two-polar decomposition of affine motion. Later on some additional constraints imposed on the affine motion are reviewed, both holonomic and non-holonomic. In particular, we concentrate on certain natural non-holonomic models of the rotation-less motion. We discuss both the usual d'Alembert model and the vakonomic dynamics. The resulting equations are quite different. It is not yet clear which model is practically better. In any case they both are different from the holonomic constraints defining the rotation-less motion as a time-dependent family of symmetric matrices of placements. The latter model seems to be non-geometric and non-physical. Nevertheless, there are certain relationships between our non-holonomic models and the polar decomposition.
Keywords: affine motion, polar and two-polar decompositions, Green and Cauchy deformation tensors, non-holonomic constraints, dynamical symmetries, d'Alembert and Lusternik variational principles, vakonomic constraints.

1 Affine constraints, geometry of the polar and two-polar decompositions

Let us begin with a short review of our earlier results concerning the mechanics of affinely-rigid body [23, 24, 25]. To be honest, some of them are also partially contained in Eringen's theory of micromorphic media, i.e., continua of infinitesimal affine bodies [8]. Later on, we developed the theory in various aspects

[9, 10, 11, 12, 14, 19, 20, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 43] and some of our results were confirmed and developed by many people [15, 16, 17, 21, 22, 40, 41, 42]. Let us also mention the papers like [3, 4, 5, 6, 18, 44]. Nevertheless, in spite of numerous applications the topic does not belong to commonly known standards, and because of this a brief repetition seems to be necessary.

Let us consider a system of material points moving in n-dimensional physical space M; we assume M to be an affine space with the linear space of translations V, endowed also with the symmetric and positively-definite metric tensor $g \in V^* \otimes V^*$. If necessary, the translation vector from $x \in M$ to $y \in M$ will be denoted by $\overrightarrow{xy}$. The material space, i.e., the set of material points will be also an affine space N of the same dimension n, with the linear space of translations U. The material metric tensor will be denoted by $\eta \in U^* \otimes U^*$, and translations vectors by $\overrightarrow{ab}$ for $a, b \in U$. As usual, we say that a mapping $\phi : N \to M$ is affine if it preserves all affine relationships, i.e., there exists a linear mapping $L[\phi] : U \to V$, denoted also as $D\phi \in L(U, V)$ such that

$$\overrightarrow{\phi(a)\,\phi(b)} = L[\phi]\,\overrightarrow{ab} \tag{1.1}$$

for any pair of material points, $a, b \in N$. If y^i, a^K are affine coordinates respectively in M and N, this means obviously that ϕ is analytically given by first-order polynomials:

$$y^i = x^i + \varphi^i{}_K a^K. \tag{1.2}$$

Obviously, this definition is valid for any, not necessarily equal dimensions of N, M. The set of all affine mappings of N onto M will be denoted by $Aff(N, M)$, and the set of all one-to-one affine mappings of N onto M is denoted by $AffI(N, M)$ (affine isomorphisms). Obviously, $AffI(N, M)$ is non-empty only if $\dim N = \dim M$, and for any $\phi \in AffI(N, M)$, $\varphi = L[\phi] \in LI(U, V)$, i.e., it is a linear isomorphism of U onto V. The groups of affine and linear isomorphisms of M and V will be denoted by $GAff(M)$, $GL(V)$. They are open subsets of $Aff(M)$, $L(V)$, i.e., of the sets of all affine and linear mappings of M and V into themselves.

Every choice of affine coordinates a^K, y^i in N, M pre-assumes two things: a choice the origins $\mathfrak{O} \in N$, $\mathfrak{o} \in M$ of coordinates in N, M and a choice of bases $(\ldots, E_A, \ldots)$, $(\ldots, e_i, \ldots)$ in U, V, or equivalently, a choice of dual bases $\left(\ldots, E^A, \ldots\right)$, $\left(\ldots, e^i, \ldots\right)$ in U^*, V^*. Then we have

$$a^K(P) = \left\langle E^K, \overrightarrow{\mathfrak{O}P} \right\rangle \ , \quad y^i(p) = \left\langle e^i, \overrightarrow{\mathfrak{o}p} \right\rangle \tag{1.3}$$

for any points $P \in N$, $p \in M$. When the constant co-moving mass distribution in N is fixed and described by positive measure μ on N, then it is natural to choose $\mathfrak{O} \in N$ as the centre of mass,

$$\int \overrightarrow{\mathfrak{O}P}\, d\mu = 0. \tag{1.4}$$

The point $\mathfrak{O}$ is uniquely defined when $m = \mu(N)$ is finite, what is physically always assumed. With such a choice of $\mathfrak{O}$, the quantities x^i in (1.2) are the current coordinates of the centre of mass in M, $\mathfrak{o}_\phi = \phi(\mathfrak{O})$. Let us stress that for any, not necessarily affine, configuration $\mathfrak{o}_\phi$ is defined by the condition

$$\int \overrightarrow{\mathfrak{o}_\phi p}\, d\mu_\phi(p) = 0, \tag{1.5}$$

where μ_ϕ denotes the ϕ-transport of the measure μ from N to M. The equality $\mathfrak{o}_\phi = \phi\left(\mathfrak{O}\right)$ holds only for affine configurations.

When the choice of $\mathfrak{O}$ is fixed as above, then the configuration space of affinely-rigid body, i.e., the manifold of affine isomorphisms of N onto M, $AffI\left(N, M\right)$ becomes canonically identified with the Cartesian product $M \times LI\left(U, V\right)$:

$$\phi \equiv \left(\phi\left(\mathfrak{O}\right), L\left[\phi\right]\right) = \left(\ldots, x^i, \ldots; \ldots, \varphi^i{}_K, \ldots\right). \tag{1.6}$$

This is the splitting of degrees of freedom into translational and internal ones.

All those concepts are purely affine and the metric tensors g, η occur only on the dynamical level. Let us mention, there are also purely affine, metric-free dynamical models [34, 35], but it is quite a different story.

The affine groups $GAff\left(M\right)$, $GAff\left(N\right)$ act on the configuration space of affine body $AffI\left(N, M\right)$ through the left and right superpositions. Namely, any $\left(\mathcal{A}, \mathcal{B}\right) \in GAff\left(M\right) \times GAff\left(N\right)$ transforms the configuration $\phi \in AffI\left(N, M\right)$ as follows:

$$\phi \to \mathcal{A} \circ \phi \circ \mathcal{B}. \tag{1.7}$$

Obviously, the action of $GAff\left(M\right)$ does commute with that of $GAff\left(N\right)$. They are respectively the spatial and material transformation groups.

Describing affine configurations as in (1.6) we can represent the actions (1.7) of $\mathcal{A} \in GAff(M)$, $\mathcal{B} \in GAff(N)$ as follows:

$$\mathcal{A} \in GAff(M): \quad (x, \varphi) \to \left(\mathcal{A}(x), L\left[\mathcal{A}\right]\varphi\right), \tag{1.8}$$

$$\mathcal{B} \in GAff(N): \quad (x, \varphi) \to \left(t\left[\varphi \cdot \overrightarrow{\mathfrak{O}\mathcal{B}\left(\mathfrak{O}\right)}\right], \varphi L\left[\mathcal{B}\right]\right), \tag{1.9}$$

where for any $v \in V$, $u \in U$, the symbols $t[v]$, $t[u]$ denote translation operations in M and N, i.e., such affine transformations of M and N that

$$\overrightarrow{xt[v](x)} = v, \qquad \overrightarrow{at[u](a)} = u \tag{1.10}$$

for any $x \in M$, $a \in N$. If, after the material origin $\mathfrak{O} \in N$ is fixed, $\mathcal{B}$ is identified with $(B, b) \in GL\left(U\right) \underset{\sim}{\times} U$ (semi-direct product), then (1.9) becomes

$$(B, b): \quad (x, \varphi) \to \left(t\left[\varphi b\right](x), \varphi B\right). \tag{1.11}$$

Analytically, (1.8) and (1.9)/(1.11) are respectively given by

$$\left(\ldots, x^i \ldots; \ldots \varphi^j{}_K, \ldots\right) \to \left(\ldots, A^i{}_m x^m + a^i, \ldots; \ldots A^j{}_m \varphi^m{}_K, \ldots\right), \tag{1.12}$$

$$\left(\ldots, x^i \ldots; \ldots \varphi^j{}_K, \ldots\right) \to \left(\ldots, x^i + \varphi^i{}_M x^M, \ldots; \ldots \varphi^j{}_L B^L{}_K, \ldots\right). \qquad (1.13)$$

The structural difference between spatial (Eulerian) and material (Lagrangian) transformations is easily seen here. Let us observe that $GL(V)$, $GL(U)$ act also on the manifold $Q_{int} = LI(U,V)$ of internal/relative degrees of freedom through the obvious formulas:

$$\varphi \to A\varphi B\ , \qquad (A,B) \in GL(V) \times GL(U). \qquad (1.14)$$

Obviously, this action is non-effective and corresponding kernel is given by the subgroup:

$$\left\{\left(\lambda Id_V, \lambda^{-1} Id_U\right) : \lambda \in \mathbb{R} \setminus \{0\}\right\} \subset GL(V) \times GL(U). \qquad (1.15)$$

The subgroups of $GL(V)$, $GL(U)$ and those of $GAff(M)$, $GAff(N)$ act in a natural way on the internal and total configuration spaces $Q_{int} = LI(U,V)$, $Q = Aff(N,M) \simeq M \times Q_{int}$. Let us mention a few most important of them: orthogonal groups $O(V,g)$, $O(U,\eta)$ their rotation subgroups $SO(V,g)$, $SO(U,\eta)$, special linear groups $SL(V)$, $SL(U)$ or one-dimensional dilatation subgroups $Dil(V) = \{\lambda Id_V : \lambda \in \mathbb{R} \setminus \{0\}\}$, $Dil(U) = \{\lambda Id_W : \lambda \in \mathbb{R} \setminus \{0\}\}$. In the total configuration space Q, when translational degrees of freedom are taken into account, those groups are semi-directly extended by translations $T(M) \simeq V$, $T(N) \simeq U$ to the corresponding affine subgroups: Euclidean $E\left(M,g\right)$, $E\left(N,\eta\right)$, isochoric $SAff(M)$, $SAff(N)$ and dilatations/translations $Dil(M)$, $Dil(N)$. The meaning of symbols is obvious. Let us only remind a few definitions. $A \in O(V,g)$, $B \in O(U,\eta)$, $\varphi \in O(U,\eta;V,g)$ when they preserve the metric tensors, thus,

$$g_{ij} = g_{kl} A^k{}_i A^l{}_j, \quad \eta_{AB} = \eta_{CD} B^C{}_A B^D{}_B, \quad \eta_{AB} = g_{ij} \varphi^i{}_A \varphi^j{}_B. \qquad (1.16)$$

$A \in SO(V,g)$, $B \in SO(U,\eta)$, when not only $|\det A| = |\det B| = 1$, but just $\det A = \det B = 1$. When orientations ρ, ω in U, V are fixed and $\det \varphi = 1$ in some orthonormal positively oriented bases in U, V, then we say that $\varphi \in SO\left(U,\eta,\rho;V,g,\omega\right)$ when $\det \varphi = 1$ and that $\varphi \in O\left(U,\eta;V,g\right)$ when $|\det \varphi| = 1$ Similarly, we say that $A \in SL(V)$, $B \in SL(U)$ when $\det A = \det B = 1$ but without orthogonality condition (1.16). And similarly $\varphi \in SL\left(U,\rho;V,\omega\right)$ when $\det \varphi = 1$ in some positively oriented bases and that $\varphi \in UL(U,V)$ when $|\det \varphi| = 1$. If $|\det A| = |\det B| = 1$ we say that, A, B are unimodular and write that $A \in UL(V)$, $B \in UL(U)$. If $\det A = \det B = 1$ we say that they are special linear.

Lie algebras of $GL^+(V)$, $GL^+(U)$, the proper (positive-determinants) subgroups of $GL(V)$, $GL(U)$ are isomorphic with the commutator Lie algebras of all linear mappings $GL(V)' \simeq L(V)$, $GL(U)' \simeq L(U)$. And Lie algebras $SO(V,g)'$, $SO(U\eta)'$ consist respectively of g- and η-skew-symmetric elements of $L(V)$, $L(U)$:

$$a^i{}_j = -a_j{}^i = -g_{jk} g^{il} a^k{}_l, \quad b^A{}_B = -b_B{}^A = -\eta_{BC} \eta^{AD} b^C{}_D. \qquad (1.17)$$

Lie algebras $SL(V)', SL(U)'$ consist of trace-less linear mappings:

$$Tr\ a = a^i{}_i = 0, \quad Tr\ b = b^K{}_K = 0. \tag{1.18}$$

As usual, when dealing with group-theoretic degrees of freedom, it is convenient to use non-holonomic Lie-algebraic velocities. In the case of affine systems we shall use the term "affine velocity"; Eringen referred to them as "gyrations". The spatial and material affine velocities of internal motion are given by

$$\Omega = \frac{d\varphi}{dt}\varphi^{-1} \in L(V), \quad \widehat{\Omega} = \varphi^{-1}\frac{d\varphi}{dt} = \varphi^{-1}\Omega\varphi \in L(U). \tag{1.19}$$

Besides the usual velocity of translational motion, $v^i = dx^i/dt$, one uses also its co-moving representation:

$$\widehat{v} = \varphi^{-1}v, \quad \widehat{v}^A = \left(\varphi^{-1}\right)^A{}_i v^i. \tag{1.20}$$

When gyroscopic constraints of metrically-rigid motion are imposed,

$$\varphi \in SO(U, \eta, \rho; V, g, \omega), \tag{1.21}$$

then Ω, $\widehat{\Omega}$ are respectively g- and η-skew-symmetric, i.e., they satisfy (1.17) when substituted instead a, b. This is the alternative, "anholonomic" representation of those holonomic constraints.

When the body is incompressible, $\det\varphi = 1$, then Ω, $\widehat{\Omega}$ are trace-less, i.e., they satisfy (1.18).

Gyroscopic and isochoric constraints are, obviously, holonomic. Geometrically this has to do with the fact that their affine velocities are elements of the commutator Lie subalgebras of $L(V)$, $L(U)$. There are, however, another interesting cases of non-holonomic constraints of spatially and materially rotation-less motion. In the first case Ω is g-symmetric, in the second one $\widehat{\Omega}$ is η-symmetric, i.e., respectively,

$$\Omega^i{}_j = \Omega_j{}^i = g_{jk}g^{il}\Omega^k{}_l, \quad \widehat{\Omega}^A{}_B = \widehat{\Omega}_B{}^A = \eta_{BC}\eta^{AD}\widehat{\Omega}^C{}_D. \tag{1.22}$$

Let us observe that unlike in the holonomic gyroscopic constraints, the two conditions (1.22) are non-equivalent and describe different non-holonomic constraints. Namely, the first, i.e., spatial, condition in (1.22) is materially represented by

$$\widehat{\Omega}^A{}_B = G_{BC}G^{AD}\widehat{\Omega}^C{}_D, \tag{1.23}$$

where G_{KL} are components the Green deformation tensor,

$$G_{KL} = g_{ij}\varphi^i{}_K\varphi^j{}_L, \tag{1.24}$$

and G^{KL} represent its contravariant inverse,

$$G^{KM}G_{ML} = \delta^K{}_L, \quad G^{KL} \neq \eta^{KM}\eta^{LN}G_{NM}. \tag{1.25}$$

Similarly, the second, i.e., material, equation in (1.22) is in the spatial language equivalent to

$$\Omega^{i}{}_{j} = C^{ik} C_{jm} \Omega^{m}{}_{k}, \tag{1.26}$$

where C_{ij} are components of the Cauchy deformation tensor,

$$C_{ij} = \eta_{AB} \left(\varphi^{-1}\right)^{A}{}_{i} \left(\varphi^{-1}\right)^{B}{}_{j}, \tag{1.27}$$

and C^{ij} are coordinates of its contravariant inverse:

$$C^{ik} C_{kj} = \delta^{i}{}_{j}, \quad C^{ij} \neq g^{ik} g^{jl} C_{kl}. \tag{1.28}$$

The inequalities in (1.25), (1.28) show that the thoughtless use of the kernel-index convention may be misleading. Having in U, V two metric-like tensors G, η and C, g one can construct two mixed tensors:

$$\widehat{G}^{A}{}_{B} := \eta^{AC} G_{CB}, \quad \widehat{C}^{i}{}_{j} := g^{ik} C_{kj} \tag{1.29}$$

and the family of scalars, e.g.,

$$I_k = Tr\left(\widehat{G}^{k}\right) = Tr\left(\widehat{C}^{-k}\right). \tag{1.30}$$

Those scalars are invariant under the action (1.14) of the subgroup $O(V,g) \times O(U,\eta) \subset GL(V) \times GL(U)$. They are basic orthogonal deformation invariants of φ. According to the Cayley-Hamilton theorem, there are only n independent invariants, e.g., those corresponding to $k = 1, \ldots, n$; any other invariant is their function.

Deformation invariants tell us how strongly the body is stretched/contracted, but they do not contain any information about the spatial or material orientation of the stretching. This information is encoded in directions of the main axes of deformation tensors G, C. More precisely, let $L_a[\varphi]$, $R_a[\varphi]$, $a = 1, \ldots, n$, be orthonormal basic eigenvectors of $\widehat{C}[\varphi]$, $\widehat{G}[\varphi]$:

$$\widehat{C} L_a = \lambda_a^{-1} L_a, \quad \widehat{G} R_a = \lambda_a R_a, \tag{1.31}$$

$$g\left(L_a, L_b\right) = g_{ij} L^{i}{}_{a} L^{j}{}_{b} = \delta_{ab} = \eta_{CD} R^{C}{}_{a} R^{D}{}_{b} = \eta\left(R_a, R_b\right). \tag{1.32}$$

Their dual covectors $L^a[\varphi] \in V^*$, $R^a[\varphi] \in U^*$ satisfy

$$C[\varphi] = \sum_a \lambda_a^{-1}[\varphi] L^a[\varphi] \otimes L^a[\varphi], \tag{1.33}$$

$$G[\varphi] = \sum_a \lambda_a[\varphi] R^a[\varphi] \otimes R^a[\varphi]. \tag{1.34}$$

It is convenient to introduce the symbols Q^a, q^a,

$$Q^a = \exp\left(q^a\right) = \sqrt{\lambda_a}. \tag{1.35}$$

The ordered bases $L = (\ldots, L_a, \ldots)$, $R = (\ldots, R_a, \ldots)$ are identified with isomorphisms $L : \mathbb{R}^n \rightarrow V$, $R : \mathbb{R}^n \rightarrow U$ and their dual co-bases $L^{-1} =$

$(\dots, L^a, \dots)$, $R^{-1} = (\dots, R^a, \dots)$ may be interpreted as the inverse isomorphisms $L^{-1} : V \to \mathbb{R}^n$, $R^{-1} : U \to \mathbb{R}^n$. The diagonal matrix with diagonal entries Q^a, $Diag\,(\dots, Q^a, \dots)$, is identified with an isomorphism $D : \mathbb{R}^n \to \mathbb{R}^n$. And finally, the isomorphism $\varphi : U \to V$ may be represented as:

$$\varphi = LDR^{-1}. \tag{1.36}$$

In this way affine configurations φ are identified with the triplets consisting of two gyroscopic configurations of metrically-rigid bodies L, R, and of the system of n material points on $\mathbb{R}$ (deformation invariants).

For any pair of linear bases, e.g., $(\dots, R_a[\varphi], \dots)$ in U and $(\dots, L_a[\varphi], \dots)$ in V, there exists exactly one linear mappings $U[\varphi]$ of U onto V such that

$$L_a[\varphi] = U[\varphi]R_a[\varphi], \quad a = 1, \dots, n. \tag{1.37}$$

When the bases are respectively η- and g-orthonormal, then $U[\varphi] \in O(U, \eta; V, g)$, and φ may be expressed as

$$\varphi = U[\varphi]A[\varphi] = B[\varphi]U[\varphi], \tag{1.38}$$

where the linear mappings $A[\varphi] \in GL(U)$, $B[\varphi] \in GL(V)$ are symmetric in the η- and g-sense and positively-definite. Obviously, by the positive definiteness we mean

$$\eta\left(A\left[\varphi\right]z, z\right) > 0, \quad g\left(B\left[\varphi\right]\omega, \omega\right) > 0 \tag{1.39}$$

and by the symmetry

$$\eta\left(A\left[\varphi\right]u, v\right) = \eta\left(u, A\left[\varphi\right]v\right), \quad g\left(B\left[\varphi\right]x, y\right) = g\left(x, B\left[\varphi\right]y\right) \tag{1.40}$$

for non-vanishing $z \in U$, $\omega \in V$ and for any $u, v \in U$, $x, y \in V$.

Unlike the two-polar splitting (1.36) the both versions of the polar splitting (1.38) are unique. And, obviously, $A[\varphi]$, $B[\varphi]$ are related to each other by the $U[\varphi]$-similarity :

$$A[\varphi] = U[\varphi]^{-1}B[\varphi]U[\varphi]. \tag{1.41}$$

Analytically, in the matrix language L, R, U are orthogonal, D is diagonal positive, and A, B are symmetric positively definite matrices.

Transformations (1.7), (1.14) act on affine velocities according to the obvious rules:

$$\Omega \to A\Omega A^{-1}, \quad \widehat{\Omega} \to B^{-1}\widehat{\Omega}B. \tag{1.42}$$

According to the same rules orthogonal transformations act on the g- and η-skew-symmetric angular velocities.

In analogy to non-holonomic affine velocities, one introduces their dual affine spin quantities,

$$\Sigma = \varphi P, \quad \widehat{\Sigma} = P\varphi = \varphi^{-1}\Sigma\varphi, \tag{1.43}$$

where P denotes the system of canonical momenta $P^{A}{}_{i}$ conjugate to $\varphi^{i}{}_{A}$, $\Sigma^{i}{}_{j}$, $\widehat{\Sigma}^{A}{}_{B}$ are momentum mappings of the transformation group (1.14). In other

words, they are Hamiltonian generators of this group. Their Poisson brackets correspond to the structure constants of $GL(V)$, $GL(U)$:

$$\begin{aligned}\left\{\Sigma^{i}{}_{j},\Sigma^{k}{}_{l}\right\} &= \delta^{i}{}_{l}\Sigma^{k}{}_{j}-\delta^{k}{}_{j}\Sigma^{i}{}_{l}, &(1.44)\\ \left\{\widehat{\Sigma}^{A}{}_{B},\widehat{\Sigma}^{C}{}_{D}\right\} &= \delta^{C}{}_{B}\widehat{\Sigma}^{A}{}_{D}-\delta^{A}{}_{D}\widehat{\Sigma}^{C}{}_{B}, &(1.45)\\ \left\{\Sigma^{i}{}_{j},\widehat{\Sigma}^{A}{}_{B}\right\} &= 0. &(1.46)\end{aligned}$$

And, obviously, the following holds:

$$\left\{\Sigma^{i}{}_{j},\varphi^{k}{}_{A}\right\}=\delta^{k}{}_{j}\varphi^{i}{}_{A},\quad \left\{\widehat{\Sigma}^{A}{}_{B},\varphi^{k}{}_{C}\right\}=\delta^{A}{}_{C}\varphi^{k}{}_{B}. \tag{1.47}$$

Transformations (1.7), (1.14) act on $\Sigma^{i}{}_{j}$, $\widehat{\Sigma}^{A}{}_{B}$ just like on Ω, $\widehat{\Omega}$ (1.42):

$$\Sigma\rightarrow A\Sigma A^{-1},\quad \widehat{\Sigma}\rightarrow B^{-1}\widehat{\Sigma}B. \tag{1.48}$$

The same may be done with the gyroscopic degrees of freedom of the two-polar and polar decompositions. The $\mathbb{R}^{n}$-comoving angular velocity $\widehat{\chi}^{a}{}_{b}$ of the L-top and the V-spatial representation $\chi^{i}{}_{j}$ are given by

$$\widehat{\chi}^{a}{}_{b}=\left\langle L^{a},\frac{dL_{b}}{dt}\right\rangle=L^{a}{}_{i}\frac{dL^{i}{}_{b}}{dt}, \tag{1.49}$$

$$\chi^{i}{}_{j}=\frac{dL^{i}{}_{a}}{dt}L^{a}{}_{j},\quad \chi=\widehat{\chi}^{a}{}_{b}L_{a}\otimes L^{b}. \tag{1.50}$$

And similarly, the $\mathbb{R}^{n}$-comoving and U-spatial components of the angular velocity of the R-top, $\widehat{\vartheta}^{a}{}_{b}$, $\vartheta^{K}{}_{L}$ are given by

$$\widehat{\vartheta}^{a}{}_{b}=\left\langle R^{a},\frac{dR_{b}}{dt}\right\rangle=R^{a}{}_{K}\frac{dR^{K}{}_{b}}{dt}, \tag{1.51}$$

$$\vartheta^{K}{}_{L}=\frac{dR^{K}{}_{a}}{dt}R^{a}{}_{L},\quad \vartheta=\widehat{\vartheta}^{a}{}_{b}R_{a}\otimes R^{b}. \tag{1.52}$$

Obviously, the "comoving" angular velocities $\widehat{\chi}^{a}{}_{b}$, $\widehat{\vartheta}^{a}{}_{b}$ are δ-antisymmetric, while the "spatial" ones, $\chi^{i}{}_{j}$, $\vartheta^{K}{}_{L}$, are respectively g- and η-antisymmetric. The dual spins conjugate to $\widehat{\chi}^{a}{}_{b}$, $\widehat{\vartheta}^{a}{}_{b}$, $\chi^{i}{}_{j}$, $\vartheta^{K}{}_{L}$ are also skew-symmetric matrices, denoted respectively by $\widehat{\rho}^{a}{}_{b}$, $\widehat{\tau}^{a}{}_{b}$, $\rho^{i}{}_{j}$, $\tau^{K}{}_{L}$. When canonical momenta conjugate to q^{a} are denoted by p_{a}, then the duality relations have the form:

$$\begin{aligned}\left\langle\left(\widehat{\rho},\widehat{\tau},p\right),\left(\widehat{\chi},\widehat{\vartheta},\dot{q}\right)\right\rangle &= \left\langle\left(\rho,\tau,p\right),\left(\chi,\vartheta,\dot{q}\right)\right\rangle\\ &= p_{a}\dot{q}^{a}+\frac{1}{2}Tr\left(\widehat{\rho}\widehat{\chi}\right)+\frac{1}{2}Tr\left(\widehat{\tau}\widehat{\vartheta}\right)\\ &= p_{a}\dot{q}^{a}+\frac{1}{2}Tr\left(\rho\chi\right)+\frac{1}{2}Tr\left(\tau\vartheta\right). &(1.53)\end{aligned}$$

It is clear that ρ, τ are Hamiltonian generators of the transformation groups,

$$\varphi\rightarrow A\varphi B^{-1},\qquad A\in SO(V,g),\quad B\in SO(U,\eta), \tag{1.54}$$

so, they are equal respectively to the spin and minus vorticity. Let us remind that the spin and vorticity are doubled g- and η-skew-symmetric parts of $\Sigma^i{}_j$, $\widehat{\Sigma}^A{}_B$. When we use the polar splitting (1.38), then the gyroscopic U-motion is characterized by angular velocity in the co-moving and spatial representations, respectively:

$$\widehat{\omega} = U^{-1}\frac{dU}{dt}, \quad \omega = \frac{dU}{dt}U^{-1} = U\widehat{\omega}U^{-1}. \tag{1.55}$$

2 Kinetic energy, equations of motion, additional constraints

It may be easily shown that the kinetic energy of the classical affine body is given by

$$T = T_{tr} + T_{int} = \frac{m}{2}g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{1}{2}g_{ij}\frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt}J^{AB}, \tag{2.1}$$

where $m \in \mathbb{R}$ and $J \in U \otimes U$ are constant inertial parameters of affine degrees of freedom,

$$m = \int\limits_N d\mu(a), \qquad J^{AB} = \int\limits_N a^A a^B d\mu(a). \tag{2.2}$$

Therefore, m is the total mass of the body and J^{AB} is the quadrupole momentum of the mass distribution, algebraically equivalent to the co-moving inertial tensor. Let us repeat that, following (1.4), the dipole momentum vanishes:

$$\int a^A d\mu(a) = 0. \tag{2.3}$$

And the higher multipoles, although non-vanishing, do not contribute to the affine motion.

The kinetic energy (2.1) is invariant under the action (1.14) of $O(V, g) \times O(U, J^{-1}) \subset GL(V) \times GL(U)$, and of course under the action of translations. It is important that because of the essential dependence of its metric tensor on g, J, it fails to be invariant under the total action of affine groups $GAff(M)$, $GL(V)$, $GAff(N)$, $GL(U)$ .After substitution of the polar decomposition (1.38) to (2.1), it becomes

$$\begin{aligned} T_{int} \quad &= \quad \frac{1}{2}\eta_{KL}\frac{dA^K{}_A}{dt}\frac{dA^L{}_B}{dt}J^{AB} + \eta_{KL}\widehat{\omega}^K{}_C A^C{}_A\frac{dA^L{}_B}{dt}J^{AB} \\ &+ \quad \frac{1}{2}\eta_{KL}\widehat{\omega}^K{}_C\widehat{\omega}^L{}_D A^C{}_A A^D{}_B J^{AB}. \end{aligned} \tag{2.4}$$

The first term represents the kinetic energy of deformative vibrations, the second one is the Coriolis coupling between deformative and rotational motion, and the third term describes the centrifugal coupling of rotations and deformations. The lower-case indices of the third term are contracted with the A-deformed

inertial tensor, $A^{C}{}_{A}A^{D}{}_{B}J^{AB}$. A similar formula holds for the second of the polar decompositions (1.38). When we use the purely analytical language and orthonormal coordinates, $\eta_{KL} = \delta_{KL}$, then the formula (2.4) may be written in the following brief matrix form:

$$T_{int} = \frac{1}{2}Tr\left(J\left(\frac{dA}{dt}\right)^{2}\right) + Tr\left(AJ\frac{dA}{dt}\widehat{\omega}\right) - \frac{1}{2}Tr\left(AJA\widehat{\omega}^{2}\right). \qquad (2.5)$$

Substituting the two-polar decomposition (1.36) and its by-products (1.49), (1.50), (1.51), (1.52) to (2.1), we obtain a rather complicated formula:

$$\begin{aligned} T_{int} &= \frac{1}{2}Tr\left(\left(\frac{dD}{dt}\right)^{2}R^{-1}JR\right) \\ &+ \frac{1}{2}Tr\left(\frac{dD}{dt}\widehat{\chi}DR^{-1}JR\right) - \frac{1}{2}Tr\left(D\widehat{\chi}\frac{dD}{dt}R^{-1}JR\right) \\ &+ \frac{1}{2}Tr\left(\widehat{\vartheta}D\frac{dD}{dt}R^{-1}JR\right) - \frac{1}{2}Tr\left(\frac{dD}{dt}D\widehat{\vartheta}R^{-1}JR\right) \\ &- \frac{1}{2}Tr\left(D\widehat{\chi}^{2}DR^{-1}JR\right) - \frac{1}{2}Tr\left(\widehat{\vartheta}D^{2}\widehat{\vartheta}R^{-1}JR\right) \\ &+ \frac{1}{2}Tr\left(\widehat{\vartheta}D\widehat{\chi}DR^{-1}JR\right) + \frac{1}{2}Tr\left(D\widehat{\chi}D\widehat{\vartheta}R^{-1}JR\right). \qquad (2.6) \end{aligned}$$

It is seen that the complication is due to the term $R^{-1}JR$. And indeed, the to polar splitting is computationally optimal in the special case of inertially isotropic body, when

$$J^{AB} = I\eta^{AB} \qquad (2.7)$$

and the kinetic energy is invariant under the material orthogonal group $O(U,\eta)$. Then

$$\left(R^{-1}JR\right)^{ab} = I\delta^{ab} \qquad (2.8)$$

and the formula (2.6) simplifies to

$$T_{int} = \frac{I}{2}Tr\left(\left(\frac{dD}{dt}\right)^{2}\right) + ITr\left(D\widehat{\chi}D\widehat{\vartheta}\right) - \frac{I}{2}Tr\left(D^{2}\widehat{\chi}^{2}\right) - \frac{I}{2}Tr\left(D^{2}\widehat{\vartheta}^{2}\right). \quad (2.9)$$

It is clear that combining appropriately $\widehat{\chi}$, $\widehat{\vartheta}$ we can avoid interference terms. This is explicitly seen when instead of the usual kinetic formulas like (2.1), (2.4), (2.5), (2.9) one uses their canonical forms based on the Legendre transformation, like, e.g.,

$$p_{i} = mg_{ij}\frac{dx^{j}}{dt}, \quad p^{A}{}_{i} = g_{ij}\frac{d\varphi^{j}{}_{B}}{dt}J^{AB} \qquad (2.10)$$

in the case of the usual, velocity-independent potentials. Then instead of (2.1) we obtain

$$\mathcal{T} = \mathcal{T}_{tr} + \mathcal{T}_{int} = \frac{1}{2m}g^{ij}p_{i}p_{j} + \frac{1}{2}\left(J^{-1}\right)_{AB}p^{A}{}_{i}p^{B}{}_{j}g^{ij}, \qquad (2.11)$$

where, obviously, g^{ij} is the contravariant inverse of g_{ij}, and $\left(J^{-1}\right)_{AB}$ is the covariant inverse of J^{AB},

$$g^{ik}g_{kj} = \delta^{i}{}_{j}, \quad \left(J^{-1}\right)_{AC} J^{CB} = \delta_{A}{}^{B}. \tag{2.12}$$

Obviously, if there is a velocity-dependence in the potential, then the formula for $\mathcal{T}$ is more complicated. For example, the presence of magnetic fields results in the configuration-dependent translational gauging of canonical momenta. But in a moment we are not interested in such details. Making use of the duality (1.53) we can write the Hamiltonian form of (2.9) as follows:

$$\mathcal{T}_{int} = \frac{1}{2I}\sum_{a} P_{a}{}^{2} + \frac{1}{8I}\sum_{a,b}\frac{\left(M^{a}{}_{b}\right)^{2}}{\left(Q^{a} - Q^{b}\right)^{2}} + \frac{1}{8I}\sum_{a,b}\frac{\left(N^{a}{}_{b}\right)^{2}}{\left(Q^{a} + Q^{b}\right)^{2}}, \tag{2.13}$$

where the quantities $M^{a}{}_{b}$, $N^{a}{}_{b}$ are given by

$$M^{a}{}_{b} := -\widehat{\rho}^{a}{}_{b} - \widehat{\tau}^{a}{}_{b}, \quad N^{a}{}_{b} := \widehat{\rho}^{a}{}_{b} - \widehat{\tau}^{a}{}_{b}, \tag{2.14}$$

Q^{a} are given by (1.35), and P_{a} are their conjugate momenta,

$$P_{a} = p_{a} \exp\left(-q^{a}\right). \tag{2.15}$$

It is seen that in (2.13) one deals with a kind of "diagonalization" of the expression for $\mathcal{T}_{int}$.

Equations of affine motion may be derived on the basis of the variational principle for the following Lagrangian:

$$L = T - V(x, \varphi), \tag{2.16}$$

where T is given by (2.1), or in the Hamiltonian terms:

$$\frac{dF}{dt} = \{F, H\}, \tag{2.17}$$

where H is the Hamiltonian corresponding to L, and F runs over a set of $2n(n+1)$ independent phase-space functions. But independently of this variational framework, they may be derived in general, on the basis of d'Alembert principle. According to this principle, equations of affine motion are obtained from the general equation of motion of the underlying system of material points by taking the monopole and dipole moments of the balance laws for the linear momentum. The points is that the original equations should be modified by introducing the reactions responsible for maintaining of the constraints. The reactions themselves do not vanish, but their monopole and dipole moments do so. Because of this, the effective, free of unspecified reactions equations of affine motion have the form of the balance laws for the total linear momentum and the total affine momentum (hypermomentum, affine spin):

$$\frac{dk^{i}}{dt} = F^{i}, \quad \frac{dK^{ij}}{dt} = \frac{d\varphi^{i}{}_{A}}{dt}\frac{d\varphi^{j}{}_{B}}{dt}J^{AB} + N^{ij} \tag{2.18}$$

with the following meaning of symbols:

$$k = \int v(P) d\mu(P), \tag{2.19}$$

$$K = \int \overrightarrow{\mathfrak{o}_\phi(P)\phi(P)} \otimes v(P) d\mu(P), \tag{2.20}$$

$$N = \int \overrightarrow{\mathfrak{o}_\phi(P)\phi(P)} \otimes F(P) d\mu(P). \tag{2.21}$$

Therefore, k is the total linear momentum, K is the total dipole moment of the momentum distribution (taken with respect to the instantaneous position of the centre of mass) and N is the total dipole momentum (affine torque), also related to the centre of mass instantaneous position. Reaction forces responsible for affine constraints are automatically cancelled in (2.19)–(2.21). The formulas (2.19)–(2.21) are general, but their affine versions are just

$$k^i = m\frac{dx^i}{dt}, \quad K^{ij} = \varphi^i{}_A \frac{d\varphi^j{}_B}{dt} J^{AB} \tag{2.22}$$

in the sense of symbols (1.2).

Let us observe that the usual Legendre transformation (for velocity-independent, usual potentials) identifies those kinematical quantities with Hamiltonian ones p_i, $\Sigma^i{}_j$ up to the index position, inessential in Cartesian coordinates:

$$p_i = g_{ij}k^j, \quad \Sigma^i{}_j = K^{im}g_{mj}. \tag{2.23}$$

It also convenient to use the co-moving representation of linear momentum, affine spin and affine moment of forces:

$$\widehat{p}^A = \left(\varphi^{-1}\right)^A{}_i k^i, \tag{2.24}$$

$$\widehat{K}^{AB} = \left(\varphi^{-1}\right)^A{}_i \left(\varphi^{-1}\right)^B{}_j K^{ij}, \tag{2.25}$$

$$\widehat{N}^{AB} = \left(\varphi^{-1}\right)^A{}_i \left(\varphi^{-1}\right)^B{}_j N^{ij}. \tag{2.26}$$

As it was said, the general, non-variational equations of motion are given by (2.18), (2.22). Let us quote a few equivalent forms also based on (2.18) with substituted (2.22), like, e.g.,

$$\frac{dk^i}{dt} = F^i, \quad \frac{dK^{ij}}{dt} = \Omega^i{}_m K^{mj} + N^{ij}. \tag{2.27}$$

Let us also observe that one can write:

$$\frac{dK^{ij}}{dt} = N^{ij} + 2\frac{\partial T_{int}}{\partial g_{ij}}. \tag{2.28}$$

In particular, for the doubled skew-symmetric part of K^{ij}, i.e., for the angular momentum, we have

$$\frac{dS^{ij}}{dt} = \frac{dK^{ij}}{dt} - \frac{dK^{ji}}{dt} = N^{ij} - N^{ji} = \mathcal{N}^{ij}, \tag{2.29}$$

where the right-hand side denotes the usual torque. If N^{ij} is symmetric, in particular vanishing, this becomes the usual conservation of angular momentum. The purely Lagrangian, i.e., U-based form of equations of motion, may be formulated as follows:

$$\frac{d\widehat{k}^A}{dt} = -\widehat{k}^B \left(J^{-1}\right)_{BC} \widehat{K}^{CA} + \widehat{F}^A, \tag{2.30}$$

$$\frac{d\widehat{K}^{AB}}{dt} = -\widehat{K}^{AC} \left(J^{-1}\right)_{CD} \widehat{K}^{DB} + \widehat{N}^{AB}. \tag{2.31}$$

Expressing these equations in terms of kinematical quantities one obtains

$$m\frac{d\widehat{v}^A}{dt} = -m\widehat{\Omega}^A{}_B\widehat{v}^B + \widehat{F}^A, \tag{2.32}$$

$$\frac{d\widehat{\Omega}^B{}_C}{dt}J^{CA} = -\widehat{\Omega}^B{}_D\widehat{\Omega}^D{}_C J^{CA} + \widehat{N}^{AB}. \tag{2.33}$$

The explicit form of equations of motion reads:

$$m\frac{d^2x^i}{dt^2} = F^i\left(x^j, \frac{dx^j}{dt}; \varphi^k{}_A, \frac{d\varphi^k{}_A}{dt}; t\right), \tag{2.34}$$

$$\varphi^i{}_A\frac{d^2\varphi^j{}_B}{dt^2}J^{AB} = N^{ij}\left(x^m, \frac{dx^m}{dt}; \varphi^k{}_C, \frac{d\varphi^k{}_C}{dt}; t\right). \tag{2.35}$$

The assumed non-singularity of matrices $\left[\varphi^i{}_A\right]$, $\left[J^{AB}\right]$ in principle enables one to solve the second equation with respect to second derivatives $d^2\varphi^i{}_A/dt^2$, expressing them through dynamical variables. Nevertheless, the form (2.35) is more convenient, because it is geometrically suited to the nature of our problem, in particular to additional constraints which may be imposed on the affine motion. This follows from the fact that in affinely-rigid behaviours the formula for the power of forces is given by

$$\mathcal{P} = \mathcal{P}_{tr} + \mathcal{P}_{int} = F^j g_{ij} v^i + N^{jk} g_{ik}\Omega^i{}_j = F_i v^i + N^j{}_i\Omega^i{}_j. \tag{2.36}$$

And if for F^j, N^{jk} one substitutes reactions maintaining constraints, this expression vanishes; only the external given forces contribute here. Similarly, if we subject the general affine motion to some natural group-theoretical constraints, in a consequence of which, e.g., Ω does belong to some Lie subalgebra of $L(V)$ or to some other linear subspace of clear algebraic meaning, then the effective reaction-free system of equations of motion consists of some natural subspace of (2.35) and, obviously, of the explicit description of constraints. This would not be the case if we used the form of (2.35) solved with respect to the second derivatives of $\varphi^j{}_B$. Let us quote a few convincing examples of both holonomic and non-holonomic constraints.

2.1 Metrically rigid motion

It consists in that both the mappings ϕ, φ are metrical isometries, therefore, all distances and angles are preserved during the motion, so that the following

holds:

$$\eta_{AB} = g_{ij}\varphi^i{}_A\varphi^j{}_B. \tag{2.37}$$

This explicit holonomic representation implies that $\Omega^i{}_j$, $\widehat{\Omega}^A{}_B$ are respectively g- and η-skew-symmetric during any admissible motion,

$$\Omega^i{}_j = -\Omega_j{}^i = -g_{ja}g^{ib}\Omega^a{}_b, \quad \widehat{\Omega}^A{}_B = -\widehat{\Omega}_B{}^A = -\eta_{BC}\eta^{AD}\widehat{\Omega}^C{}_D. \tag{2.38}$$

Therefore, (2.35) is not valid any longer, instead the following equations with unspecified reaction torques hold:

$$\frac{dK^{ij}}{dt} = \frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt}J^{AB} + N^{ij} + N_R{}^{ij}\,, \tag{2.39}$$

or equivalently

$$\varphi^i{}_A\frac{d^2\varphi^j{}_B}{dt^2}J^{AB} = N^{ij} + N_R{}^{ij}\,. \tag{2.40}$$

But the d'Alembert principle tells us that the power $\mathcal{P}_R$ of gyroscopic reactions vanishes on every $\Omega^i{}_j$ compatible with constraints, i.e., g-skew-symmetric:

$$\mathcal{P}_R = N_R{}^i{}_j\Omega^j{}_i = N_R{}^{ij}\,\Omega_{ji} = 0 \tag{2.41}$$

if (2.38) holds. Therefore,

$$N_R{}^{ij} = N_R{}^{ji}\,, \quad N_R{}^i{}_j = N_R{}_j{}^i = g_{ja}g^{ib}\,N_R{}^a{}_b. \tag{2.42}$$

But this simply means that the reaction-free gyroscopic equations of motion consist of the skew-symmetric part of (2.39) or (2.40) with substituted (2.37), e.g.,

$$\varphi^i{}_A\frac{d^2\varphi^j{}_B}{dt^2}J^{AB} - \varphi^j{}_A\frac{d^2\varphi^i{}_B}{dt^2}J^{AB} = N^{ij} - N^{ji} = \mathcal{N}^{ij}. \tag{2.43}$$

In other words, any parametrization of the isometry manifold ("Euler angles", "rotation vectors", etc.) may be safely substituted to (2.43). The torque $\mathcal{N}^{ij} = N^{ij} - N^{ji}$ becomes the function of those parameters and we obtain the system of $n\,(n-1)\,/2$ independent of equations (2.43) imposed on $n\,(n-1)\,/2$ parameters. It is clear that this system is the balance law for spin:

$$\frac{dS^{ij}}{dt} = \frac{d}{dt}\left(K^{ij} - K^{ji}\right) = \mathcal{N}^{ij}. \tag{2.44}$$

It becomes the spin conservation when the torque $\mathcal{N}^{ij}$ does vanish, i.e., when N^{ij} is symmetric. One can as well rewrite (2.44) in Lagrangian terms:

$$\begin{aligned}\frac{d\widehat{S}^{AB}}{dt} = \frac{d}{dt}\left(\widehat{K}^{AB} - \widehat{K}^{BA}\right) &= \widehat{K}^{BC}\left(J^{-1}\right)_{CD}\widehat{K}^{DA} \\ &- \widehat{K}^{AC}\left(J^{-1}\right)_{CD}\widehat{K}^{DB} + \widehat{\mathcal{N}}^{AB},\end{aligned} \tag{2.45}$$

where

$$\widehat{\mathcal{N}}^{AB} = \widehat{N}^{AB} - \widehat{N}^{BA}. \tag{2.46}$$

This may be easily expressed in terms of co-moving affine velocity:

$$\frac{d\widehat{\Omega}^{B}{}_{C}}{dt}J^{CA}-\frac{d\widehat{\Omega}^{A}{}_{C}}{dt}J^{CB}=\widehat{\Omega}^{A}{}_{D}\widehat{\Omega}^{D}{}_{C}J^{CB}-\widehat{\Omega}^{B}{}_{D}\widehat{\Omega}^{D}{}_{C}J^{CA}+\widehat{\mathcal{N}}^{AB}. \tag{2.47}$$

Those are Euler equations. When the co-moving inertial tensor is spherical, i.e., when

$$J=\frac{I}{2}\eta, \tag{2.48}$$

then the non-dynamical terms on the right-hand side of (2.47) do vanish, and we obtain simply

$$I\frac{d\widehat{\Omega}^{AB}}{dt}=I\frac{d\widehat{\Omega}^{A}{}_{C}}{dt}\eta^{CB}=\widehat{\mathcal{N}}^{AB}. \tag{2.49}$$

2.2 Shape-preserving motion

Now the shape of the body is preserved, but not necessarily its size, so that

$$g_{ij}\varphi^{i}{}_{A}\varphi^{j}{}_{B}=\lambda\eta_{AB}, \tag{2.50}$$

where λ denotes the time-dependent coefficient. Then, in analogy to the previous example, the d'Alembert principle tells us that the reactions-free equations of the constrained motion consist of the skew-symmetric part and the trace of (2.18) or (2.35):

$$\frac{dS^{ij}}{dt} = \frac{d}{dt}\left(K^{ij}-K^{ji}\right)=\mathcal{N}^{ij}=N^{ij}-N^{ji}, \tag{2.51}$$

$$\frac{dK^{i}{}_{i}}{dt} = \frac{d}{dt}\left(g_{ij}K^{ij}\right)=g_{ij}\frac{d\varphi^{i}{}_{A}}{dt}\frac{d\varphi^{j}{}_{B}}{dt}J^{AB}+g_{ij}N^{ij}=2T+N^{i}{}_{i}, \tag{2.52}$$

when written in a few independent forms. Using directly the representation in terms of coordinates, we obtain that

$$\varphi^{i}{}_{A}\frac{d^{2}\varphi^{j}{}_{B}}{dt^{2}}J^{AB}-\varphi^{j}{}_{A}\frac{d^{2}\varphi^{i}{}_{B}}{dt^{2}}J^{AB} = \mathcal{N}^{ij}=N^{ij}-N^{ji}, \tag{2.53}$$

$$g_{ij}\varphi^{i}{}_{A}\frac{d^{2}\varphi^{j}{}_{B}}{dt^{2}}J^{AB} = g_{ij}N^{ij}. \tag{2.54}$$

2.3 Incompressible affine motion

Incompressibility (isochoric motion) means that

$$\frac{d}{dt}\det\left[\varphi^{i}{}_{A}\right]=0. \tag{2.55}$$

This condition is well defined, although $\det\left[\varphi^{i}{}_{A}\right]$ is not a scalar but scalar density with respect to both spatial and material coordinate transformations.

The identity

$$\frac{d}{dt}\det\left[\varphi^{i}{}_{A}\right]=\det\left[\varphi^{j}{}_{B}\right]\left(\varphi^{-1}\right)^{A}{}_{i}\frac{d\varphi^{i}{}_{A}}{dt}=0 \tag{2.56}$$

is equivalent to

$$Tr\,\Omega = Tr\,\widehat{\Omega} = \left(\varphi^{-1}\right)^{A}{}_{i}\frac{d\varphi^{i}{}_{A}}{dt} = 0. \tag{2.57}$$

D'Alembert principle implies that reactions N_R which keep these constraints, being dual to the subspace of all trace-less matrices,

$$\mathcal{P}_R = N_R{}^{i}{}_{j}\Omega^{j}{}_{i} = 0, \quad \Omega^{k}{}_{k} = 0, \tag{2.58}$$

are proportional to the identity mapping,

$$N_R{}^{i}{}_{j} = \lambda\delta^{i}{}_{j}, \quad N_R{}^{ij} = \lambda g^{ij}. \tag{2.59}$$

Therefore, the effective reactions-free system of equations of motion is given by the trace-less part of the original balance law for K^{ij}, i.e., explicitly

$$\varphi^{i}{}_{A}\frac{d^2\varphi^{j}{}_{B}}{dt^2}J^{AB} - \frac{1}{n}g_{ab}\varphi^{a}{}_{A}\frac{d^2\varphi^{b}{}_{B}}{dt^2}J^{AB}g^{ij} = N^{ij} - \frac{1}{n}g_{ab}N^{ab}g^{ij}. \tag{2.60}$$

This is a system of $\left(n^2-1\right)$ independent equations of motion imposed on $\left(n^2-1\right)$ independent parameters of $\varphi^{i}{}_{A}$ (cf. (2.55)).

2.4 Spatially rotation-less motion

We have seen above that the concept of the purely rotational degrees of freedom is well defined and correctly formulated: simply the mappings ϕ, φ are isometries. It is not so with the opposite concept of rotation-free, i.e., purely deformative configurations. The first, naive idea would be to base this concept on the polar decomposition, either in left or right version. So, rotation-free configurations would be ones given by the purely deformative factor in the polar decomposition. Therefore, depending on whether one deals with the U-left or U-right version of (1.38), we would say that φ is purely deformative when it coincides with its η- or g-symmetric part $A[\varphi]$ or $B[\varphi]$. However, this would be incorrect. The more incorrect would be attempts of introducing the pure deformation on the basis of the two-polar decomposition (1.36). There are a few deep geometric reasons for that. First of all, the symmetric mappings $A[\varphi]$, $B[\varphi]$ do not describe any configurations at all. It is only mappings from U to V that may be used as a model of the configuration, neither the linear automorphisms of U nor those of V. Without fixing some standard element of the manifold of isometries $O\left(U,\eta;V,g\right)$ we cannot identify automorphisms of the material or physical spaces with any mappings of U on to V. So, even from this relatively naive point of view, the symmetric mappings do not describe configurations. But there are also other arguments. Namely, even if we "forget" about the above fact and simply proceed with the $\mathbb{R}^n$-model of space and body manifolds, using the elements of $GL(n,\mathbb{R})$ as $LI(U,V)$, it is still so that the symmetric matrices do not form a Lie group. Therefore, the corresponding absence of rotation is not an equivalence relation because of the transitivity failure. If some configurations A, B are mutually non-rotated in the polar sense, i.e., they are related by the

symmetric matrix $Sym(A, B)$, and if so are B, C in the sense of being obtained from each other by the action of some $Sym(B, C)$, then in general A, C are not connected by a symmetric matrix. It is well known that the symmetric matrices do not form a Lie group or Lie algebra. The product $Sym(A, B)\,Sym(B, C)$ in general is not symmetric; instead it splits into the multiplication of some symmetric matrix and some nontrivial isometry. So, certainly, being related by a symmetric matrix is not an equivalence relation, and the concept of mutually rotation-free configurations is not correct. But there are well-defined rotation-less motions. We say that the motion $\mathbb{R} \ni t \to \varphi(t) \in LI(U, V)$ is spatially rotation-less when Ω is g-symmetric:

$$\Omega^{i}{}_{j} - \Omega_{j}{}^{i} = \Omega^{i}{}_{j} - g_{jk} g^{il} \Omega^{k}{}_{l} = 0. \tag{2.61}$$

And similarly, we say that it is materially rotation-less when $\widehat{\Omega}$ is η-symmetric:

$$\widehat{\Omega}^{A}{}_{B} - \widehat{\Omega}_{B}{}^{A} = \widehat{\Omega}^{A}{}_{B} - \eta_{BC} \eta^{AD} \widehat{\Omega}^{C}{}_{D} = 0. \tag{2.62}$$

The symmetry of Ω or $\widehat{\Omega}$ is just the natural complementary concept of their antisymmetry in rigid motion. And therefore, this is a proper definition of the rotation-less behaviour, just behaviour, not configuration. The point is that the symmetric matrices do not form a Lie algebra. On the contrary, they are anti-Lie algebras in the sense that their commutators are respectively g- and η-skew-symmetric:

$$\begin{aligned} \left[Sym\left(L\left(V\right), g\right), Sym\left(L\left(V\right), g\right)\right] &= Asym\left(L\left(V\right), g\right) \simeq SO\left(V, g\right)', & (2.63)\\ \left[Sym\left(L\left(U\right), \eta\right), Sym\left(L\left(U\right), \eta\right)\right] &= Asym\left(L\left(U\right), \eta\right) \simeq SO\left(U, \eta\right)'. & (2.64) \end{aligned}$$

This is an interesting example of non-holonomic constraints, in a sense different than the classical constraints of slide-free motion. Nevertheless, some relationship with the usual non-holonomic problems of non-sliding motion still seems to exist in certain hypothetical applications. Let us consider, e.g., an affine motion of a small inclusion or droplet suspension in very viscous fluid. It is natural to expect that the surface friction may be an obstacle against rotations. And then probably the effective constraints of rotation-less motion may appear.

Let us stress some circumstance. Namely, the holonomic gyroscopic constraints may be written alternatively in two apparently non-holonomic forms:

$$\Omega^{i}{}_{j} + \Omega_{j}{}^{i} = 0, \quad \widehat{\Omega}^{A}{}_{B} + \widehat{\Omega}_{B}{}^{A} = 0. \tag{2.65}$$

They are mutually equivalent. On the other side, the two versions of non-holonomic constraints (2.61) and (2.62) are non-equivalent. Namely, the g-symmetry of Ω is equivalent to the $\widehat{G}$-symmetry of $\widehat{\Omega}$ where, as usual, $\widehat{G}_{AB}$ denotes the Green deformation tensor, so that (2.61) is identical with

$$G_{AC} \widehat{\Omega}^{C}{}_{B} - G_{BC} \widehat{\Omega}^{C}{}_{A} = 0. \tag{2.66}$$

In a moment we are unable to answer the question concerning the details of this relationship and the possible fields of physical applications. From a perhaps

naive point of view, it seems to be so that it is rather the Euler symmetry (2.61) that seems to be applicable to description of the affine motion of suspensions in viscous fluids.

In any case, it is an interesting and rather new problem to discuss the structure of equations of motion subject to rotation-less non-holonomic constraints. Again the d'Alembert principle shows the advantage of the K-balance form of equations. Namely, the effective, reactions-free equations are given by the symmetric part of the balance laws,

$$\varphi^{i}{}_{A}\frac{d^2\varphi^{j}{}_{B}}{dt^2}J^{AB}+\varphi^{j}{}_{A}\frac{d^2\varphi^{i}{}_{B}}{dt^2}J^{AB}=N^{ij}+N^{ji}, \tag{2.67}$$

together with the algebraically substituted constraints (2.61). The right-hand side of (2.67) depends only on given forces and is free of reactions. The Lagrange form of (2.67) is given by

$$\frac{d\widehat{\Omega}^{B}{}_{C}}{dt}J^{CA}+\frac{d\widehat{\Omega}^{A}{}_{C}}{dt}J^{CB}=-\widehat{\Omega}^{B}{}_{D}\widehat{\Omega}^{D}{}_{C}J^{CA}-\widehat{\Omega}^{A}{}_{D}\widehat{\Omega}^{D}{}_{C}J^{CB}+\widehat{N}^{AB}+\widehat{N}^{BA}, \tag{2.68}$$

where $\widehat{\Omega}$ is subject to (2.66).

2.5 Materially rotation-less motion

It is also non-holonomic and somehow related to the spatially rotation-less situation, nevertheless, in our opinion it is a bit less intuitive. Now the material gyration is assumed to be η-symmetric, i.e., (2.62) is assumed to hold. The effective, reaction-free equations of motion may be written as follows:

$$\begin{aligned}&\frac{d\widehat{K}^{AC}}{dt}\widehat{\mathcal{D}}_{C}{}^{B}+\frac{d\widehat{K}^{BC}}{dt}\widehat{\mathcal{D}}_{C}{}^{A}=\widehat{N}^{AC}\widehat{\mathcal{D}}_{C}{}^{B}+\widehat{N}^{BC}\widehat{\mathcal{D}}_{C}{}^{A}\\&-\widehat{K}^{AM}\left(J^{-1}\right)_{MN}\widehat{K}^{NC}\widehat{\mathcal{D}}_{C}{}^{B}-\widehat{K}^{BM}\left(J^{-1}\right)_{MN}\widehat{K}^{NC}\widehat{\mathcal{D}}_{C}{}^{A},\end{aligned} \tag{2.69}$$

where, obviously, $\widehat{K}^{AB}$ are co-moving components of K^{ij}, and

$$\widehat{K}^{AB}=\widehat{\Omega}^{B}{}_{C}J^{AC},\quad \widehat{\mathcal{D}}_{A}{}^{B}=G_{AC}\eta^{CB}. \tag{2.70}$$

These equations are much more complicated than those for the spatially rotation-less motion. Namely, their non-dynamical terms depend on the Green tensor, therefore, also on the configuration φ. The Euler form is also complicated:

$$\varphi^{i}{}_{A}\frac{d^2\varphi^{b}{}_{B}}{dt^2}J^{AB}g_{bc}C^{cj}+\varphi^{j}{}_{A}\frac{d^2\varphi^{b}{}_{B}}{dt^2}J^{AB}g_{bc}C^{ci}=N^{ib}g_{bc}C^{cj}+N^{jb}g_{bc}C^{ci}, \tag{2.71}$$

where

$$C^{ab}=\varphi^{a}{}_{A}\varphi^{b}{}_{B}\eta^{AB} \tag{2.72}$$

is the inverse Cauchy tensor.

3 Dynamical symmetries of affine motion

The non-holonomic constraints of rotation-less motion, i.e., the above examples described in subsections 2.4 and 2.5, are really exceptional and in a sense surprising within the realm of constrained affine motion. Let us stress to avoid some easy misunderstandings: they are really non-holonomic and have nothing to do with apparently suggestive constraints of the type that φ in (1.38) is symmetric. Moreover, our analysis above shows that such a formulation would be inconsistent, just because of fixing some of infinitely possible isometries $U[\varphi]$. And, let us repeat, the symmetric matrices do not form a Lie group. The symmetry of Ω or $\widehat{\Omega}$ leads to certain equations satisfied by $U[\varphi]$, $A[\varphi]$, $B[\varphi]$ in (1.38), but these equations are differential, not algebraic ones. The g-symmetry of Ω or η-symmetry of $\widehat{\Omega}$ are the only natural counterparts of their antisymmetry in rigid motion. And in any case, they are geometrically interesting special cases of constraints, worth to be investigated from the very point of view of purely analytical mechanics.

It is interesting to "solve" the constraints equations (2.61), i.e., to "parametrize" somehow the manifold of non-holonomic constraints. The best candidates are suggested by the polar decomposition (1.38). Let us remind that $U[\varphi]$ is an isometry and that $A[\varphi]$ is η-symmetric, thus,

$$\eta_{AB} = g_{ij} U[\varphi]^{i}{}_{A} U[\varphi]^{j}{}_{B}, \tag{3.1}$$

$$\eta_{AC} A^{C}{}_{B} = \eta_{BC} A^{C}{}_{A}, \quad \eta_{AC} \frac{d}{dt} A^{C}{}_{B} = \eta_{BC} \frac{d}{dt} A^{C}{}_{A}, \tag{3.2}$$

and the co-moving angular velocity of the U-rotator, $\widehat{\omega} \in O\left(U, \eta\right)' \subset L\left(U\right)$ is given by (1.55) and is, obviously, η-skew-symmetric:

$$\eta_{AC} \widehat{\omega}^{C}{}_{B} = -\eta_{BC} \widehat{\omega}^{C}{}_{A}. \tag{3.3}$$

Substituting those conditions to the definition (1.19) of the affine velocity Ω, we obtain after easy calculations the conclusion that

$$\widehat{\omega}^{A}{}_{B} = \frac{1}{2} \left(\left(A^{-1}\right)^{A}{}_{C} \frac{dA^{C}{}_{B}}{dt} - \frac{dA^{A}{}_{C}}{dt} \left(A^{-1}\right)^{C}{}_{B} \right). \tag{3.4}$$

Therefore, the angular velocity of the U-rotator equals the half of the commutator of two algebraically independent instantaneous quantities A^{-1}, dA/dt:

$$\widehat{\omega} = \frac{1}{2} \left[A^{-1}, \frac{dA}{dt} \right]. \tag{3.5}$$

In any case, this quantity in general does not vanish and this reflects the non-holonomic character of our constraints of non-rotational motion. It is something different than the constancy of U, i.e., the vanishing of $\widehat{\omega}$. Making use of the polar decomposition (1.38) and gyroscopic angular velocity (1.55), we can, a bit formally, write down the constraints equations (3.5) in the following Pfaff form:

$$U^{-1} dU - \frac{1}{2} A^{-1} dA + \frac{1}{2} \left(dA\right) A^{-1} = 0. \tag{3.6}$$

This system of Pfaff equations is evidently non-integrable.

Before going any further with the analysis of the constrained affine motion, let us quote a few remarks concerning the invariance problems. We are interested mainly in the internal, i.e., relative, motion and concentrate on the spatial and material rotational invariance. The second, i.e., internal, equation (2.35) is $O(V,g)$-invariant if for any its solution $t \rightarrow \varphi(t)$ and for any $A \in O(V,g)$ the motion $t \rightarrow A\varphi(t)$ is also a solution. This implies that

$$N^{ij}\left(A\varphi, A\frac{d\varphi}{dt}\right) = A^{i}{}_{k}A^{j}{}_{l}N^{kl}\left(\varphi, \frac{d\varphi}{dt}\right), \tag{3.7}$$

where we do not indicate explicitly the possible explicit time-dependence of N. But (3.7) means that the co-moving representation of N is non-sensitive with respect to the action of $A \in O(V,g)$:

$$\widehat{N}\left(A\varphi, A\frac{d\varphi}{dt}\right) = \widehat{N}\left(\varphi, \frac{d\varphi}{dt}\right). \tag{3.8}$$

This means that $\widehat{N}$ is algebraically built of the co-moving quantities $\widehat{G}$, $\widehat{\Omega}$ and any fixed material tensor $\widehat{K}$ in $\widehat{U}$:

$$\widehat{N}\left(\varphi, \frac{d\varphi}{dt}\right) = \widehat{F}\left(G, \widehat{\Omega}, \widehat{K}\right). \tag{3.9}$$

It is interesting that this form of $\widehat{N}$ implies the rotational invariance of equations of motion, but it does not imply the conservation of spin, i.e., internal angular momentum. Spin is conserved only if $\widehat{N}$ is a symmetric tensor, just like in mechanics of micropolar or micromorphic continua. Let us mention in particular that the Green-Ostrogradskij theorem implies that in orthonormal Cartesian coordinates the affine momentum of forces is proportional to the mean value of Cauchy stress tensor in the medium:

$$N^{ij} = -\int \sigma^{ij}. \tag{3.10}$$

In the case if hyperelastic bodies, both continuous and discrete, N^{ij}, $\widehat{N}^{AB}$ are automatically symmetric. Indeed, the condition for the potential energy

$$\mathcal{V}\left(A\varphi\right) = \mathcal{V}\left(\varphi\right), \quad A \in O\left(V,g\right), \tag{3.11}$$

implies that $\mathcal{V}$ is algebraically built of the Green deformation tensor:

$$\mathcal{V}\left(\varphi\right) = W\left(G\left[\varphi\right], \widehat{K}\right), \tag{3.12}$$

where K again denotes any state-independent tensor in U. And then one can show immediately that

$$\widehat{N}\left(\varphi\right)^{AB} = 2\frac{\partial W}{\partial G_{AB}} = \widehat{N}\left(\varphi\right)^{BA}, \quad N^{ij} = N^{ji}, \tag{3.13}$$

therefore, spin is a conserved quantity.

This was about the invariance under the left-hand side action of $O(V,g)$ on internal/relative degrees of freedom. Let us now ask what are conditions of the invariance under the right-hand side action of material orthonormal group $O(U,\eta)$. One can easily show that for any solution $t \to \varphi(t)$ of (2.35) and for any $B \in O(U,\eta)$ the right-rotated motion $t \to \varphi(t)B$ is a solution too when the following holds:

$$\varphi^{i}{}_{K}\frac{d^{2}\varphi^{j}{}_{L}}{dt^{2}}B^{K}{}_{C}B^{L}{}_{D}J^{CD} = N^{ij}\left(\varphi B, \frac{d\varphi}{dt}B\right). \tag{3.14}$$

Unlike in the case of spatial isotropy, this implies two conditions — the internal and dynamical ones:

$$J = I\eta, \quad N\left(\varphi B, \frac{d\varphi}{dt}B\right) = N\left(\varphi, \frac{d\varphi}{dt}\right). \tag{3.15}$$

The second conditions in (3.15) implies that N depends on the mechanical state $(\varphi, d\varphi/dt)$ through the pair (C,Ω) and any fixed, i.e., state-independent tensors K in V:

$$N\left(\varphi, \frac{d\varphi}{dt}\right) = H\left(C,\Omega,K\right). \tag{3.16}$$

For hyperelastic bodies with the right-invariant potential energy,

$$\mathcal{V}\left(\varphi\right) = \mathcal{V}\left(\varphi B\right), \qquad B \in O\left(U,\eta\right), \tag{3.17}$$

the following holds:

$$\mathcal{V}\left(\varphi\right) = W\left(C\left[\varphi\right],K\right), \tag{3.18}$$

where again K denotes any system of state-independent tensors in V.

An important question appears as to when the dynamics of an affine hyperelastic body is simultaneously isotropic in space and matter. Obviously, this holds only when (3.12), (3.15), (3.18) are simultaneously satisfied. Therefore, the inertial tensor is spherical, $J = I\eta$, and the potential energy $\mathcal{V}$ depends on φ only through the deformation invariants, e.g., through the quantities I_k (1.30), or any of alternative expressions like λ_a, q^a, Q^a or other used in the two-polar decomposition like (1.35). Therefore,

$$\mathcal{V}\left(\varphi\right) = F\left(I_1,\ldots,I_n\right) = G\left(\lambda_1,\ldots,\lambda_n\right), \tag{3.19}$$

where, obviously, G is invariant under the group $S^{(n)}$ of all permutations of its arguments.

It is clear that according to the general rules of Hamiltonian mechanics, in the potential motion of the affinely-rigid body the reactions-free affine moment of forces is given by

$$N^{i}{}_{j} = -\varphi^{i}{}_{A}\frac{\partial\mathcal{V}}{\partial\varphi^{j}{}_{A}}, \quad N^{ij} = -\varphi^{i}{}_{A}\frac{\partial\mathcal{V}}{\partial\varphi^{k}{}_{A}}g^{kj}, \tag{3.20}$$

and similar formulas hold for the co-moving representation.

The relationships (2.24)–(2.26) imply that in a general, not necessarily hyperelastic, case equations of internal motion are simultaneously spatially and materially isotropic, when $J = I\eta$ and $\widehat{N}$ is given by (3.9) with $\widehat{K} = \eta$ or, equivalently, by (3.16) with $K = g$. For example, in a rather academic elastic, but not necessarily hyperelastic, situation using the Cayley-Hamilton theorem one can show that

$$\widehat{N}_{el}{}^{A}{}_{B} = \sum_{a=1}^{n} B_a \left(I_1, \ldots, I_n\right) \left(\widehat{G}^{a-1}\right)^{A}{}_{B}, \tag{3.21}$$

where $\widehat{N}_{el}{}^{A}{}_{B}$, $\widehat{G}^{A}{}_{B}$ are components of $\widehat{N}_{el}$, $\widehat{G}$ with the η-lowered index B:

$$\widehat{N}_{el}{}^{A}{}_{B} = \widehat{N}_{el}{}^{AC} \eta_{CB}, \quad \widehat{G}^{A}{}_{B} = \widehat{G}^{AC} \eta_{CB}, \tag{3.22}$$

and the scalar coefficients B_a in expansion (3.21) depend on deformation invariants. One can show that in the hyperelastic case, when the potential $\mathcal{V}\left(I_1, \ldots, I_n\right)$ does exist, the coefficients are given by the following derivatives:

$$B_a = -2a \frac{\partial \mathcal{V}}{\partial I_a}. \tag{3.23}$$

Obviously, the physical utility of elastic but not hyperelastic models is rather doubtful, nevertheless it must be admitted for the completeness of the theory.

Another example of a doubly isotropic model is one concerning the isotropic internal friction in continuum droplet. The viscous stress tensor is given in a linear approximation by

$$\sigma^{ij}_{visc} = 2\nu d^{ij} + \left(\zeta - \frac{2\nu}{n}\right) g_{ab} d^{ab} g^{ij}, \tag{3.24}$$

where the constants ν, ζ are viscosity coefficients and d^{ij} is the deformation rate tensor. In the case of affine body it is given by

$$d^{ij} = \frac{1}{2}\left(\Omega^{ij} + \Omega^{ji}\right), \quad \Omega^{ij} = \Omega^{i}{}_{k} g^{kj}. \tag{3.25}$$

Then, making use of the obvious formula (3.10) we obtain that

$$N^{ij}_{visc} = -V_0 \sqrt{\frac{\det\left[g_{ij}\right]}{\det\left[\eta_{AB}\right]}} \det\left[\varphi^{i}{}_{A}\right] \left(\nu\left(\Omega^{ij} + \Omega^{ji}\right) + \left(\zeta - \frac{2\nu}{n}\right) \Omega^{k}{}_{k} g^{ij}\right), \tag{3.26}$$

where V_0 denotes the standard (Lagrangian) volume of the affine body. This agrees with the formula (3.16) with $K = g$.

The total viscoelastic and doubly-isotropic moment of forces is given by

$$N^{ij} = N^{ij}_{el} + N^{ij}_{visc} \tag{3.27}$$

with the separate terms like (3.21) (3.26).

Those were interesting and instructive examples of the doubly isotropic (spatially and materially) internal forces N^{ij}. It is also interesting to find a description of more general isotropic forces, adapted to certain special parametrizations of the configuration space. In particular, some possibilities of partial separation of variables, or rather their subsystems, appear then. First of all, let us begin with the polar splitting (1.38), more precisely, with its first form where the orthogonal term $U[\varphi]$ stands on the left-hand side. As mentioned above, gyroscopic kinetics is described by the co-moving angular velocity $\widehat{\omega} = U^{-1}dU/dt$, whereas deformation (together with its orientation with respect to the body) is represented by the η-symmetric and positive factor $A[\varphi]$ in (1.38). As usual, the following tensors with η-shifted indices will be employed:

$$J^{A}{}_{B} = J^{AC}\eta_{CB}, \qquad \widehat{N}^{A}{}_{B} = \widehat{N}^{AC}\eta_{CB}, \tag{3.28}$$

$$G^{A}{}_{B} = \eta^{AC}G_{CB}, \qquad A^{KL} = A^{K}{}_{M}\eta^{ML}. \tag{3.29}$$

It is clear that

$$\widehat{\Omega} = A^{-1}\widehat{\omega}A + A^{-1}\frac{dA}{dt} = A^{-1}\left(\widehat{\omega} + \frac{dA}{dt}A^{-1}\right)A \tag{3.30}$$

and

$$N^{ij} = \varphi^{i}{}_{C}\varphi^{j}{}_{D}\widehat{N}^{CD} = U^{i}{}_{K}U^{j}{}_{L}A^{K}{}_{C}A^{L}{}_{D}\widehat{N}^{CD}. \tag{3.31}$$

This suggests us to introduce the following quantity:

$$\overline{N}^{KL} = A^{K}{}_{C}A^{L}{}_{D}\widehat{N}^{CD}, \qquad \text{i.e.,} \qquad \overline{N} = (A \otimes A)\,\widehat{N}. \tag{3.32}$$

Just like $\widehat{N}$ itself, $\overline{N}$ is also an element of $U \otimes U$, however of a quite different nature. Namely, $\widehat{N}^{AB}$ are components of N with respect to the basis φE_{A}, $A = 1, \ldots, n$, affinely co-moving with the body. Unlike this, the quantities $\overline{N}^{AB}$ are components of N with respect to the orthonormal basis $U[\varphi]\,E_{A}$, $A = 1, \ldots, n$, co-moving with the $U[\varphi]$-gyroscope of the polar decomposition of φ.

After this substitution, our internal equations of motion, i.e., the second subsystem (2.35), become as follows:

$$\begin{aligned} A^{K}{}_{C}J^{C}{}_{D}\frac{d^{2}A^{DM}}{dt^{2}} - A^{K}{}_{C}J^{C}{}_{D}A^{D}{}_{E}\frac{d\widehat{\omega}^{EM}}{dt} - 2A^{K}{}_{C}J^{C}{}_{D}\frac{dA^{D}{}_{E}}{dt}\widehat{\omega}^{EM} & \\ + A^{K}{}_{C}J^{C}{}_{D}A^{D}{}_{E}\widehat{\omega}^{E}{}_{F}\widehat{\omega}^{FM} = \overline{N}^{KM} & \end{aligned} \tag{3.33}$$

with the convention (3.28)–(3.29) concerning the η-shift of tensor indices. Obviously, for the spatially isotropic models one is faced with some kind of partial separation of variables. Indeed, the spatial isotropy means that $\overline{N}$ is independent on the variable U. It is a function of the state quantities A, dA/dt, $\widehat{\omega}$ only. Roughly speaking, in the non-holonomic $\widehat{\omega}$-representation it is a kind of cyclic state variable. Therefore, the procedure of solving equations of motion splits into three steps:

1) (3.33) is a system of differential equations for the time dependence of quantities A, $\widehat{\omega}$.

2) Assuming that the previous step is done, we write the system of differential equations for U,

$$\frac{dU}{dt} = U\widehat{\omega}\,(t)\,. \tag{3.34}$$

Let us observe that this system is time-dependent through the time evolution of $\widehat{\omega}\,(t)$.

3) When the steps 1), 2) are performed, we construct the final solution:

$$\varphi\,(t) = U\,(t)\,A\,(t)\,. \tag{3.35}$$

Obviously, this is only the general scheme. For dynamically realistic models the steps 1), 2) as a rule, are not analytically solvable. Nevertheless, even this partial separation and a sequence of procedures is very helpful for the understanding the problem. In any case, its structure looks simpler and more adapted to operations. Let us only quote two examples corresponding to (3.21), (3.26). After substituting (3.28)–(3.29) we find respectively that

$$\overline{N}_{el} = \sum_{a=1}^{n} B_a\,(I_1 \ldots I_n)\,A^{2a}, \tag{3.36}$$

$$\begin{aligned} \overline{N}_{visc} &= -V_0 \det A \left[\nu \left(\frac{dA}{dt} A^{-1} + A^{-1} \frac{dA}{dt} \right) \right. \\ &+ \left. \left(\zeta - \frac{2\nu}{n} \right) Tr \left(\frac{dA}{dt} A^{-1} \right) \eta^{-1} \right]. \end{aligned} \tag{3.37}$$

It is a nice feature of the both formulas that $\overline{N}_{el}$ depends only on A, and $\overline{N}_{visc}$ depends only on A, dA/dt; there is no dependence on $\widehat{\omega}$. This independence is due to the fact that $\overline{N}_{visc}$ describes the internal friction. To be honest, the linear dependence of $\overline{N}_{visc}$ on the g-symmetric part of Ω is an approximation valid in the case of small internal velocities. In general, the higher powers of $\Omega^{(ij)}$ are admissible.

Both expressions (3.21), (3.26), therefore, also (3.36), (3.37), have an additional interesting feature of being isotropic simultaneously in space and material. It is natural to ask for the optimal way of expressing this fact. As expected, the most natural way consists in using the two-polar representation (1.36) and the related quantities (1.35), (1.49), (1.50), (1.51), (1.52). Then, identifying the factors L, R in (1.36) with linear mappings from $\mathbb{R}^n$ to V and U respectively, and similarly identifying $\widehat{\chi}$, $\widehat{\vartheta}$, D with linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$, we obtain

the following formulas:

$$\Omega = L\left(\widehat{\chi} + \frac{dD}{dt}D^{-1} - D\widehat{\vartheta}D^{-1}\right)L^{-1}, \tag{3.38}$$

$$\widehat{\Omega} = R\left(D^{-1}\widehat{\chi}D + D^{-1}\frac{dD}{dt} - \widehat{\vartheta}\right)R^{-1}, \tag{3.39}$$

$$\widehat{\omega} = R\left(\widehat{\chi} - \widehat{\vartheta}\right)R^{-1}, \tag{3.40}$$

obviously $\widehat{\omega}$ (1.55) is a linear mapping from U to U. These formulas are simple and suggestive. The last of them, i.e., one for $\widehat{\omega}$, is an infinitesimal expression of the obvious fact that $U = LR^{-1}$.

As mentioned in (2.23), for the potential systems the Legendre transformation relates $\Sigma^{i}{}_{j}$ to $K^{i}{}_{j} = K^{im}g_{mj}$. Similarly $\widehat{\Sigma}^{A}{}_{B}$ is related to $\widehat{K}^{AC}G_{CB}$. It is important that the second index is lowered with the help of Green deformation tensor, not with the help of the fixed material metric η_{CB}. One should not confuse $\widehat{\Sigma}^{A}{}_{B}$ with

$$\widehat{K}^{AB} = \left(\varphi^{-1}\right)^{A}{}_{i}\left(\varphi^{-1}\right)^{B}{}_{j}K^{ij}. \tag{3.41}$$

Let us quote the explicit Legendre formulas for those quantities in the special materially isotropic case $J^{AB} = I\eta^{AB}$. So, we have that

$$\Sigma = K = IL\left(D\widehat{\vartheta}D + D\frac{dD}{dt} - D^{2}\widehat{\chi}\right)L^{-1}, \tag{3.42}$$

$$\widehat{\Sigma} = \varphi^{-1}K\varphi = IR\left(\widehat{\vartheta}D^{2} + \frac{dD}{dt}D - D\widehat{\chi}D\right)R^{-1}, \tag{3.43}$$

$$\widehat{K} = \varphi^{-1}K\left(\varphi^{-1}\right)^{T} = IR\left(\widehat{\vartheta} + \frac{dD}{dt}D^{-1} - D\widehat{\chi}D\right)R^{-1}. \tag{3.44}$$

Let us mention that the corresponding spin parts, i.e., doubled skew-symmetric parts of those quantities, equal respectively to

$$S = \Sigma - \Sigma^{T} = IL\left(2D\widehat{\vartheta}D - D^{2}\widehat{\chi} - \widehat{\chi}D^{2}\right)L^{-1}, \tag{3.45}$$

$$V = \widehat{\Sigma}^{T} - \widehat{\Sigma} = IR\left(2D\widehat{\chi}D - D^{2}\widehat{\vartheta} - \widehat{\vartheta}D^{2}\right)R^{-1}, \tag{3.46}$$

$$\widehat{S} = \widehat{K} - \widehat{K}^{T} = IR\left(-D\widehat{\chi}D^{-1} - D^{-1}\widehat{\chi}D + 2\widehat{\vartheta}\right)R^{-1}. \tag{3.47}$$

Then for the doubly (spatially and materially) isotropic problems the quantities S, V are constants of motion. Unlike this, $\widehat{S}$, i.e., (3.47), is not a conserved quantity.

In (3.32) we have introduced the quantity $\overline{N} \in U \otimes U$, the components of which represented N with respect to the moving orthonormal basis $U[\varphi]E_{A}$, $A = 1, \ldots, n$. This representation enabled one to reduce equations of motion to the U-independent form (3.33). Something similar may be done for the two-polar representation. Namely, there exist an obvious analogy between (3.30)

and (3.39) in that

$$\Omega = L\widetilde{\Omega}L^{-1}, \quad \widehat{\Omega} = R\underset{\sim}{\Omega}R^{-1}, \quad \widetilde{\Omega} = D\underset{\sim}{\Omega}D^{-1}. \tag{3.48}$$

It is clear that the matrix elements of the $\mathbb{R}^n$-tensors,

$$\widetilde{\Omega} = \widehat{\chi} + \frac{dD}{dt}D^{-1} - D\widehat{\vartheta}D^{-1}, \quad \underset{\sim}{\Omega} = D^{-1}\widehat{\chi}D + D^{-1}\frac{dD}{dt} - \widehat{\vartheta}, \tag{3.49}$$

are components of Ω with respect to the orthonormal frame L_a frozen into the Cauchy gyroscope and the components of $\widehat{\Omega}$ with respect to the orthonormal frame co-moving with the Green deformation tensor. And the same representation may be introduced for any other tensor quantity, in particular for the affine moment of forces N. The mixed, contravariant-covariant representation of $\widetilde{N}$ is given by $\widetilde{N}^a{}_b$, where $\widetilde{N} = L^{-1}NL$, i.e.,

$$\widetilde{N}^a{}_b = \langle L^a, NL_b\rangle = L^a{}_i N^i{}_j L^j{}_b, \qquad N^i{}_j = N^{ik}g_{kj}. \tag{3.50}$$

Substituting the above equations to the doubly isotropic case of the internal subsystem (2.35), we obtain the following equations of motion:

$$\begin{aligned} & D\frac{d^2D}{dt^2} - D^2\frac{d\widehat{\chi}}{dt} + D\frac{d\widetilde{\vartheta}}{dt}D - 2D\frac{dD}{dt}\widehat{\chi} + 2D\widehat{\vartheta}\frac{dD}{dt} \\ & + D^2\widehat{\chi}^2 - 2D\widehat{\chi}D\widehat{\vartheta} + D\widehat{\vartheta}^2 D = \frac{1}{I}\widetilde{N}\left(D, \frac{dD}{dt}, \widehat{\chi}, \widehat{\vartheta}\right). \end{aligned} \tag{3.51}$$

The dynamical double isotropy implies that $\widetilde{N}$ depends only on the indicated variables D, dD/dt, $\widehat{\chi}$, $\widehat{\vartheta}$ but is independent of the angular variables L, R. Therefore, similarly like in (3.33), there is a partial separability of the problem (3.51):

1) Just as it was the case with (3.33), one solves the system (3.51). To be more precise, one dreams about solving this system of n^2 ordinary differential equations for the n^2 dynamical variables D, $\widehat{\chi}$, $\widehat{\vartheta}$. Some kind of rigorous solutions is possible only for the two-dimensional case $n = 2$. For higher dimensions, including the physical case $n = 3$, only some special solutions may be analytically found.

2) When the time dependence $\mathbb{R} \ni t \to \left(\widehat{\chi}(t), \widehat{\vartheta}(t)\right)$ is "known", we substitute it to the definition of angular velocities:

$$\frac{dL}{dt} = L\widehat{\chi}, \qquad \frac{dR}{dt} = \widehat{\vartheta}. \tag{3.52}$$

   Then one obtains the system of differential equations with right-hand sides explicitly dependent on time.

3) After "solving" (3.52) we substitute everything to (1.36) and obtain the final solution.

As mentioned many times above, this partial reduction (separability) of (3.33), (3.51) is rather ideal and qualitative, nevertheless, it is helpful in understanding the dynamical structure of spatially and doubly isotropic models.

4 D'Alembert and vakonomic models of rotation-less motion

Let us now discuss briefly the interesting special case of non-holonomic rotation-less constraints. This will be rather an introductory analysis; up to our knowledge nobody discussed this kind of constraints, either in the d'Alembert or vakonomic version. From some point of view the apparently exotic vakonomic form is rather simpler and more elegant [6]. It is yet rather too early to try deciding which is more physical and in what kind of problems.

Let us substitute formally the polar representation (3.4), (3.5) of (2.61) to the polar expression of the kinetic energy (2.4). Then we obtain that

$$\begin{aligned} T_{int} &= \frac{1}{8}\eta_{KL}\frac{dA^K{}_A}{dt}\frac{dA^L{}_B}{dt}^{AB} + \frac{1}{4}\eta_{KL}\left(A^{-1}\right)^K{}_D\frac{dA^D{}_C}{dt}A^C{}_A\frac{dA^L{}_B}{dt}J^{AB} \\ &+ \frac{1}{8}\eta_{KL}\left(A^{-1}\right)^K{}_E\frac{dA^E{}_C}{dt}A^C{}_A\left(A^{-1}\right)^L{}_F\frac{dA^F{}_D}{dt}A^D{}_B J^{AB}. \end{aligned} \tag{4.1}$$

After calculations this may be expressed in the following more concise form:

$$\begin{aligned} T_{int} &= \frac{1}{8}\eta_{KL}\left(A^{-1}\right)^K{}_E\left(A^{-1}\right)^L{}_F\left(A^E{}_C\frac{dA^C{}_A}{dt}\right. \\ &+ \left.\frac{dA^E{}_C}{dt}A^C{}_A\right)\left(A^F{}_D\frac{dA^D{}_B}{dt} + \frac{dA^F{}_D}{dt}A^D{}_B\right)J^{AB}. \end{aligned} \tag{4.2}$$

The variational derivative of T_{int} with respect to the symmetric tensor

$$A_{AB} = \eta_{AC}A^C{}_B = A_{BA} \tag{4.3}$$

is given by

$$\begin{aligned} \left.\frac{\delta T_{int}}{\delta A_{AB}}\right|_{symm} &= -\frac{1}{4}\frac{d^2}{dt^2}A^{(A}{}_L J^{B)L} - \frac{1}{4}\frac{d}{dt}\left(\left(A^{-1}\right)^{(A}{}_E J^{B)L}\frac{dA^E{}_C}{dt}A^C{}_L\right) \\ &- \frac{1}{4}\eta_{KL}\frac{d}{dt}\left(\frac{dA^K{}_E}{dt}\left(A^{-1}\right)^{L(A}A^{B)}{}_D\right)J^{ED} \\ &- \frac{1}{4}\eta_{KL}\frac{d}{dt}\left(\left(A^{-1}\right)^K{}_E\frac{dA^E{}_C}{dt}A^C{}_F\left(A^{-1}\right)^{L(A}A^{B)}{}_D\right)J^{FD} \\ &- \frac{1}{4}\eta_{KL}\frac{dA^K{}_E}{dt}\frac{dA^F{}_D}{dt}A^D{}_G\left(A^{-1}\right)^{L(A}A^{B)}{}_F J^{EG} \\ &- \frac{1}{4}\eta_{KL}\left(A^{-1}\right)^K{}_E\frac{dA^E{}_C}{dt}A^C{}_M\frac{dA^F{}_D}{dt}A^D{}_N\left(A^{-1}\right)^{L(A}A^{B)}{}_F J^{MN} \\ &+ \frac{1}{4}\eta_{KL}\frac{dA^K{}_D}{dt}\left(A^{-1}\right)^L{}_E\frac{dA^{E(A}}{dt}J^{B)D} \\ &+ \frac{1}{4}\eta_{KL}\left(A^{-1}\right)^K{}_E\frac{dA^E{}_C}{dt}A^C{}_D\left(A^{-1}\right)^L{}_F\frac{dA^{F(A}}{dt}J^{B)D}. \end{aligned} \tag{4.4}$$

When there are hyperelastic forces derivable from the potential $\mathcal{V}$ depending only on the Green deformation tensor, then equations of motion have the following form:

$$\left.\frac{\delta T_{int}}{\delta A_{AB}}\right|_{symm} = -A_{KC}\eta^{K(A}\widehat{N}^{B)C}, \tag{4.5}$$

where

$$\widehat{N}^{BC} = -\left(D\mathcal{V}\right)^{BC}. \tag{4.6}$$

In spite of their apparently complicated structure, equations (4.4) are readable. And having them solved for the time dependence of A_{AB}, we obtain from (3.4)/(3.5) the time dependence of $\widehat{\omega}$, and then, solving (in principle) (1.55) for dependence $t \to U(t)$, we finally obtain (in principle) $\varphi = UA$.

Let us mention that all tensor indices are shifted from their natural position with the help of η.

The usual d'Alembert procedure, i.e., the symmetric part of (3.33) with algebraically substituted constraints (2.61), i.e., (3.4)/(3.5), leads to the following form, less readable than (4.4), (4.5), (4.6):

$$\begin{aligned}
&J^{AB}\frac{d^2A^{B(C}}{dt^2}A^{D)}{}_A - J^A{}_B A^B{}_E\frac{d}{dt}\frac{1}{2}\left(\left(A^{-1}\right)^E{}_F\frac{d}{dt}\left(A^{F(C}\right)A^{D)}{}_A\right.\\
&\left.-\frac{d}{dt}\left(A^E{}_F\right)\left(A^{-1}\right)^{F(C}A^{D)}{}_A\right) - J^A{}_B\frac{dA^B{}_E}{dt}\left(\left(A^{-1}\right)^E{}_F\frac{d}{dt}\left(A^{F(C}\right)A^{D)}{}_A\right.\\
&\left.-\frac{d}{dt}\left(A^E{}_F\right)\left(A^{-1}\right)^{F(C}A^{D)}{}_A\right) + \frac{1}{4}J^A{}_B A^B{}_E\left(\left(A^{-1}\right)^E{}_G\frac{d}{dt}\left(A^G{}_F\right)\right.\\
&\left.-\frac{d}{dt}\left(A^E{}_G\right)\left(A^{-1}\right)^G{}_F\right)\left(\left(A^{-1}\right)^F{}_H\frac{d}{dt}\left(A^{H(C}\right)A^{D)A}\right.\\
&\left.-\frac{d}{dt}\left(A^F{}_H\right)\left(A^{-1}\right)^{H(C}A^{D)}{}_A\right) \quad = \quad \overline{N}^{(CD)}.
\end{aligned} \tag{4.7}$$

Again the η-shift of indices is meant here. The difference between (4.4)/(4.5) and (4.7) on their right-hand side is not essential, because it is only due to the A-term transformation of $\widehat{N}$ into $\overline{N}$. They may be written in a similar form in this sense. But the difference between other terms of (4.6) and (4.7) is more essential. The detailed analysis of this difference is postponed to the next paper. In any case, it is a general rule that the d'Alembert and vaconomic procedures give different equations.

Appendix: d'Alembert vs. vaconomic constraints

The problem appear more than century ago. It is well known that when the holonomic constraints

$$F_a(q) = 0, \quad a = 1, \ldots, m, \tag{4.8}$$

are imposed onto the motion of a Lagrangian dynamical system with generalized coordinates $q^1, \ldots, q^n$, then one can equivalently use the d'Alembert procedure or the restricted extremum (more precisely, stationary value) problem.

If Lagrangian is given by $L(q, \dot{q})$, then the Lusternik theorem tells us that the conditional extremum (more precisely, stationary value)

$$\delta \int L dt = 0, \quad F_a(q) = 0, \quad a = 1, \dots, m, \tag{4.9}$$

is given by the functions of time satisfying equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = R_i, \quad F_a(q) = 0, \tag{4.10}$$

where

$$R_i = \frac{d\mu^a}{dt}\omega_{ai}, \qquad \omega_{ai} = \frac{\partial F_a}{\partial q^i}, \qquad \frac{dF_a}{dt} = \omega_{ai}(q)\frac{dq^i}{dt}. \tag{4.11}$$

And those are exactly d'Alembert equations with the multipliers

$$\lambda^a = \frac{d\mu^a}{dt}. \tag{4.12}$$

The same formulas for reaction forces R_i hold also for non-variational, e.g., dissipative dynamical models:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_i, \tag{4.13}$$

where Q_i are non-variational generalized forces. Then as well we have that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_i + \lambda^a\frac{\partial F_a}{\partial q^i}, \quad F_a(q) = 0. \tag{4.14}$$

So, there was a natural temptation to expect something similar for systems with non-holonomic constraints, for simplicity linear in velocities,

$$\omega_{ai}(q)\frac{dq^i}{dt} = 0, \tag{4.15}$$

but without the intergrability assumption

$$\omega_{ai}\frac{\partial F_a}{\partial q^i} = 0, \tag{4.16}$$

i.e., without the vanishing of exterior differentials:

$$\frac{\partial \omega_{ai}}{\partial q^j} - \frac{\partial \omega_{aj}}{\partial q^i} \neq 0 \tag{4.17}$$

But it turned out in contrary: d'Alembert procedure gives again the equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_i + \lambda^a\omega_{ai}, \quad \omega_{ai}(q)\frac{dq^i}{dt} = 0, \tag{4.18}$$

with reactions coefficients λ^a to be eliminated. But the Lusternik theorem for

$$\delta \int L dt = 0, \qquad \omega_{ai}(q)\frac{dq^i}{dt} = 0 \tag{4.19}$$

gives something drastically else:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \frac{d\mu^a}{dt}\omega_{ai} + \mu^a \left(\frac{\partial \omega_{ai}}{\partial q^j} - \frac{\partial \omega_{aj}}{\partial q^i} \right) \dot{q}^j &= 0, &(4.20)\\ \omega_{ai}(q)\frac{dq^i}{dt} &= 0. &(4.21) \end{aligned}$$

The difference is obvious. Moreover, μ^a become a kind of dynamical variables, because they occur both by itself and their time derivatives. The problems of sliding-free rolling motion are ruled by the d'Alembert procedure. But, on the other hand, the Lusternik variational, i.e., vaconomic, procedure looks very interesting and intriguing. It gives rice to the new mathematical discipline and its applications seem to be also possible, first of all, in active control problems.

Our equations for the rolling-free affine motion in the d'Alembert and vaconomic sense are also drastically different, although as yet we are unable to express them in qualitative terms.

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